Problem Sheet 10

Problem 1

Let R be an I-adically complete ring.

(a) An idempotent is an element e with $e^2 = e$. Prove that the idempotents of R are in bijection with those of R/I.

(b) Show that every maximal ideal of R contains I.

Problem 2

Use Hensel's Lemma to prove the following.

(a) For every $a \in \mathbb{Z}$, $(a, p) = 1$, there is a unique $(p-1)$-st root of unity $\zeta \in \mathbb{Z}_p$ with $\zeta \equiv a \mod p$.

(b) Assume that $n \geq 1$ and $a \in \mathbb{Z}$ are both prime to p. Show that the equation $x^n = a$ is solvable in $\mathbb{Z}_p$ if and only if it is solvable in $\mathbb{F}_p$.

Problem 3

Let L/K be a field extension; assume it can be written as $L = \cup_{i \in \mathbb{N}} L_i$ for a sequence of subfields

$$K \subseteq L_1 \subseteq L_2 \subseteq \ldots \subseteq L_i \subseteq \ldots \subseteq L$$

with each L_i/K finite Galois.

(a) Show that the group G of field automorphisms of L/K coincides with the inverse limit

$$\lim_{i \in \mathbb{N}} \mathrm{Gal}(L_i/K).$$

(b) Endow G with the inverse limit topology w.r.t. the discrete topology on all $\mathrm{Gal}(L_i/K)$. Deduce from usual Galois theory that there is a bijection

$$\begin{array}{ccc} \{\text{intermediate fields } K \subseteq M \subseteq L\} & \cong & \{\text{closed subgroups } H \subseteq G\} \\ M & \longmapsto & \{\alpha \text{ s.th. } \alpha|_M = \mathrm{id}\} \\ \{x \text{ s.th. } \alpha(x) = x \ \forall \alpha \in H\} & \longleftarrow & H. \end{array}$$

Problem 4

Determine the Galois groups of the following field extensions.

(a) The extension $\bar{\mathbb{F}}_p/\mathbb{F}_p$, where $\bar{\mathbb{F}}_p$ denotes an algebraic closure of $\mathbb{F}_p$.

(b) The extension $\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q}$, where $\mathbb{Q}^{\mathrm{cyc}} \subseteq \mathbb{C}$ is the union of all cyclotomic extensions $\{\mathbb{Q}(\zeta_n)\}_{n \in \mathbb{N}}$.